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Three-dimensional MHD duct flows with strong transverse magnetic fields

Part 1. Obstacles in a constant area channel

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This paper is an analysis of incompressible three-dimensional flows of electrically conducting fluids under the action of transverse magnetic fields which are assumed to be sufficiently strong that the interaction parameter N $(=M^2/R) \gg 1$, where M is the Hartmann number and R is the Reynolds number. We also assume that $R \gg 1$ and R_m (magnetic Reynolds number) $\ll 1$, so that experimental verification of the theory may be possible.

The main results are: (i) when a *thick* body is placed in a parallel-sided channel with non-conducting walls the flow over it is highly dependent on the conductivity of the body, in a surprising way. If the body is *non-conducting*, there is no flow within that cylinder which circumscribes the body and is parallel to the magnetic field; outside the cylinder the flow is plane and potential and enters or leaves the surface shear layer of this cylinder at right angles. If the body is *conducting*, flow over it is possible and is of a different nature outside and inside the cylinder. (ii) When a *non-conducting flat plate* is placed in such a channel no blocking of the flow occurs. If the plate is elongated in the flow direction, the flow over it becomes identical to that calculated by Hasimoto (1960) and, if elongated at right angles to the flow, becomes identical to that calculated by Dix (1963).

Of particular interest in our analysis are the two types of layer which occur in these flows, the first being the Hartmann boundary layer, which is shown to have a controlling influence on the vorticity of the core flow in three-dimensional situations analogous to that of the Eckman layer in rotating-fluid flows. The second type, the free shear layer at the circumscribing cylinder, is of interest because of its internal structure and effect on the external flow.

1. Introduction

In the last few years, for the first time, some experiments have been performed on the effects of transverse magnetic fields on the flow of electrically conducting liquids over bodies of various shapes (e.g. Tsinober 1963). Although there has been much MHD theory on the flows over bodies in transverse fields, there has been little which can be tested in the laboratory, where the need is for a realistic theory of three-dimensional viscous flows confined by the walls of the ducts.

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The existing theory of three-dimensional flows, where the magnetic Reynolds number $R_m \ll 1$ and the Reynolds number $R \gg 1$, is inviscid and is only valid for unconfined flows; e.g. Ludford (1961) and Ludford & Singh (1963) have examined flow over various bodies when the interaction parameter $N \gg 1$, and Reitz & Foldy (1961) examined the flow over a sphere when $N \ll 1$. On the other hand, for two-dimensional flows Hunt & Leibovich (1967) were able to consider the effects of viscosity and confining walls on flows over bodies, by assuming $R_m \ll 1$,

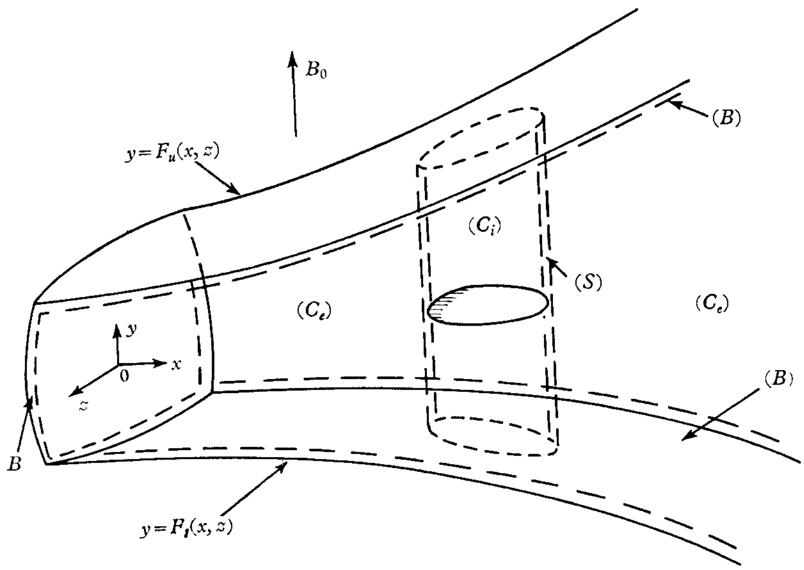

FIGURE 1. Various regions of the flow when an obstacle is placed in a duct. External and internal core flows (C_e), (C_i), boundary layers (B), and shear layer (S).

$N \gg 1$, and $R \gg 1$. The work presented here is, first, a generalization to three dimensions of the analysis of that earlier paper, and, secondly, a discussion of two subsidiary problems.

The first is to examine some of the differences between the unconfined, inviscid flows over bodies, as studied by Ludford and others, and the confined, viscous flows, attainable experimentally.

The second is to clarify the physical implications of the results of those who have studied flow over a flat plate. If Ox is in the flow direction and Oz perpendicular to Ox and the magnetic field, Oy, Dix (1963) examined the flow over the plate $y = 0, x > 0$ such that $\partial/\partial z = 0$ and found that the velocity outside the Hartmann boundary layer is that of the stream at infinity, u_∞. On the other hand Hasimoto (1960) examined the flow over a plate $y = 0$, $|z| <$ const. such that $\partial/\partial x = 0$ and found that, as a result of the 'wakes' which stem from the edges of the plate, the velocity is $\frac{1}{2}u_\infty$ above the Hartmann boundary layer. In §5 we examine the flow over a plate finite in the Ox and Oz directions and find that, as the plate's shape is changed, our solutions for the two extreme cases agree with the results of Dix and Hasimoto.

The methods used in this paper are similar to those of Hunt & Leibovich (1967), namely that the flow may be analysed in certain separate regions: the core, the boundary layers and the shear layers (see figure 1). The solutions for these various regions match each other and are consistent with the original assumptions. The novel feature of our analysis is the lack of determinacy in the zeroth-order solution (in the sense of Hunt & Leibovich), which only occurs in three-dimensional problems and can only be resolved by considering higher-order terms in the asymptotic expansions.

In §2 after studying the non-dimensional form of the equations and the general form of the solution for the core, we consider three-dimensional Hartmann boundary layers. A new jump condition is derived, which is of decisive importance in analysing three-dimensional flows, in that it determines the inviscid core flows.

In §3 we write down the solution for flow between two surfaces at $y = F_u(x, z)$ and $y = F_1(x, z)$, and in §4 we use these results to examine the flow over a symmetric body placed in a parallel-sided duct. The salient result is that there is no flow over the body, when it is *non-conducting*, but there may be if it is *conducting*. We examine the shear layers in some detail and find that they determine the external flow. In §6 we discuss these flows and those of §5 in terms of the differences between two- and three-dimensional situations; we also discuss the interesting analogies, both physical and mathematical, with non-conducting flows in a rotating environment.

2. Formulation of the problem

2.1. *Non-dimensional MHD equations*

The equations governing the steady flow of an electrically conducting fluid with uniform properties under the action of a transverse magnetic field, $\mathbf{B}_0$, may, when $R_m \ll 1$, be written (Shercliff 1965):

$$\rho(\mathbf{v}^*.\nabla)\mathbf{v}^* = -\nabla p^* + \mathbf{j}^* \times \mathbf{B}_0 + \eta\nabla^2\mathbf{v}^*, \tag{2.1}$$

$$\nabla.\mathbf{v}^* = 0, \tag{2.2}$$

$$\mathbf{j}^* = \sigma(\mathbf{E}^* + \mathbf{v}^* \times \mathbf{B}_0), \tag{2.3}$$

$$\nabla \times \mathbf{E}^* = 0, \tag{2.4}$$

$$\nabla.\mathbf{j}^* = 0, \tag{2.5}$$

where $\mathbf{v}^*$, p^*, $\mathbf{j}^*$ and $\mathbf{E}^*$ are velocity, pressure, electric current and electric field respectively, and ρ, σ and η are density, conductivity and viscosity respectively.

If we now non-dimensionalize the variables in terms of the fluid properties, a characteristic velocity U_0, and length d (typically the mean velocity in a duct and the duct width at some point), as follows:

$$\mathbf{v} = (u, v, w) = (v_x^*, v_y^*, v_z^*)/U_0, \quad p = p^*/\rho U_0^2,$$

$$\mathbf{E} = \mathbf{E}^*/U_0 B_0, \quad \mathbf{j} = \mathbf{j}^*/\sigma U_0 B_0,$$

then equations (2.1–2.5) become

$$(\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + N(\mathbf{j}\times\hat{\mathbf{y}}) + R^{-1}\nabla^2\mathbf{v}, \tag{2.6}$$

$$\boldsymbol{\nabla}.\mathbf{v} = 0, \tag{2.7}$$

$$\mathbf{j} = \mathbf{E} + \mathbf{v}\times\hat{\mathbf{y}}, \tag{2.8}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = 0, \tag{2.9}$$

$$\nabla.\mathbf{j} = 0. \tag{2.10}$$

Here $N = \sigma B_0^2 d/\rho U_0$, $R = \rho U_0 d/\eta$, and $\hat{\mathbf{y}}$ is the unit vector in the direction of $\mathbf{B}_0$ (see figure 1); the operator $\boldsymbol{\nabla}$ is now taken with respect to the non-dimensional variables x, y, z. For subsequent use we define $M = B_0 d(\sigma/\eta)^{\frac{1}{2}} = (NR)^{\frac{1}{2}}$.

From equations (2.7)–(2.9), by elminating $\mathbf{E}$, we obtain the familiar 'magnetic induction' equation

$$\boldsymbol{\nabla}\times\mathbf{j} = \partial\mathbf{v}/\partial y. \tag{2.11}$$

Equations (2.6), (2.7) and (2.10), (2.11) are the governing equations for p, $\mathbf{v}$, and $\mathbf{j}$.

We next consider the basic form of the solutions in the core regions (C) and the boundary layers (B).

2.2. *Core regions* (C)

If in C we assume that $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ are $O(1)$, then as $N\rightarrow\infty$ and $R\rightarrow\infty$, equation (2.6) becomes

$$0 = -\boldsymbol{\nabla}(p/N) + \mathbf{j}\times\hat{\mathbf{y}}. \tag{2.12}$$

It immediately follows that

$$\partial(p/N)/\partial y = 0 \tag{2.13}$$

and also that, if we take the curl,

$$\partial\mathbf{j}/\partial y = 0. \tag{2.14}$$

From (2.11) we now find

$$\partial^2\mathbf{v}/\partial y^2 = 0. \tag{2.15}$$

Equations (2.14) and (2.15) are fundamental results for large-N flows, as important as the Taylor–Proudman theorem in rotating flows, where

$$\partial\mathbf{v}/\partial y = 0 \tag{2.16}$$

in the limit of small Rossby number and large Reynolds number. It will become apparent later that the similarities in the flows over bodies in these two situations are quite marked. The paradoxical result of (2.12) that, as $N\rightarrow\infty$, the electromagnetic $\mathbf{j}\times\mathbf{B}$ force becomes *irrotational* was discussed by Hunt & Leibovich (1967).

We can now deduce the general form of the core solution by substituting the general solutions of equations (2.13)–(2.15) into (2.7,) (2.10)–(2.12). We find

$$p = Nh, \tag{2.17}$$

$$\left.\begin{aligned} u &= y(\partial\Upsilon/\partial z) - \partial h/\partial x - \partial\psi/\partial z,\\ v &= y(\partial^2 h/\partial x^2 + \partial^2 h/\partial z^2) + g,\\ w &= y(-\partial\Upsilon/\partial x) - \partial h/\partial z + \partial\psi/\partial x,\end{aligned}\right\} \tag{2.18}$$

$$j_x = \partial h/\partial z, \quad j_y = -\Upsilon, \quad j_z = -\partial h/\partial x, \tag{2.19}$$

where h, Υ, ψ and g are unknown functions of x and z *only*. To calculate these variables we have to know the boundary conditions on the walls and the upstream and downstream conditions.

2.3. *Boundary layers* (*B*)

In this section we consider the boundary layers on walls where $\mathbf{B}_0 . \mathbf{n} \neq 0$, $\mathbf{n}$ being the unit normal out of the wall. If we choose axes $O'nst$ and if we define the angle α to be that between $O'n$ and Oy, then it is well known that to zeroth order in the boundary layer, as R, $N \to \infty$, the curl of (2.6) implies†

$$\left.\begin{aligned} 0 &= -N\cos\alpha\,\partial j_t/\partial n + R^{-1}\partial^3 v_s/\partial n^3, \\ 0 &= N\cos\alpha\,\partial j_s/\partial n + R^{-1}\partial^3 v_t/\partial n^3, \end{aligned}\right\} \tag{2.20}$$

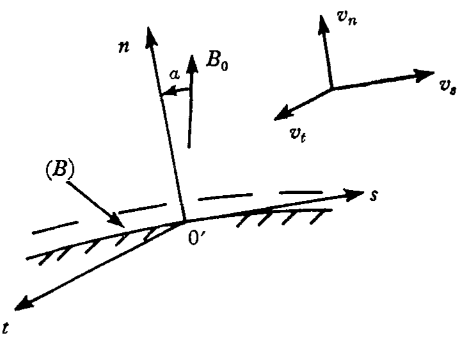

FIGURE 2. Notation for boundary-layer analysis in §2.

and the induction equation (2.11) implies

$$\left.\begin{aligned} 0 &= -\cos\alpha\,\partial v_s/\partial n + \partial j_t/\partial n, \\ 0 &= \cos\alpha\,\partial v_t/\partial n + \partial j_s/\partial n. \end{aligned}\right\} \tag{2.21}$$

If we apply the boundary condition

$$\mathbf{v} = 0 \quad \text{at} \quad n = 0$$

and denote the core values of $\mathbf{v}$ and $\mathbf{j}$ at the wall by the suffix ∞, then the basic solution is

$$\begin{pmatrix} v_s \\ v_t \end{pmatrix} = \begin{pmatrix} v_{s\infty} \\ v_{t\infty} \end{pmatrix} [1 - \exp(-Mn\cos\alpha)], \tag{2.22}$$

and
$$\begin{pmatrix} j_s \\ j_t \end{pmatrix} = \begin{pmatrix} j_{s\infty} \\ j_{t\infty} \end{pmatrix} + \begin{pmatrix} v_{t\infty} \\ -v_{s\infty} \end{pmatrix} \cos\alpha \exp(-Mn\cos\alpha). \tag{2.23}$$

Equations (2.22) and (2.7) enable us to deduce

$$v_{n\infty} = (M\cos\alpha)^{-1}(\partial v_{s\infty}/\partial s + \partial v_{t\infty}/\partial t) \tag{2.24}$$

from the condition $v_n = 0$ at $n = 0$.

† In (2.20) and (2.21) we have ignored the equations governing v_n and j_n as the variation of these two variables in the layer is of little interest.

So far the electrical properties of the wall have not entered. At the solid–fluid interface the tangential component of electric field and the normal component of current are continuous (Shercliff 1965). In an insulator there is no current, so that

$$j_n = 0 \quad \text{at} \quad n = 0. \tag{2.25}$$

In a perfect conductor at rest there is no electric field, so that

$$j_s = j_t = 0 \quad \text{at} \quad n = 0, \tag{2.26}$$

according to equation (2.8), since $v = 0$ there.

The condition

$$j_{n\infty} = M^{-1}(\partial v_{s\infty}/\partial t - \partial v_{t\infty}/\partial s) \quad \text{at a non-conducting wall} \tag{2.27}$$

now follows from equations (2.23), (2.25) and $\nabla . \mathbf{j} = 0$. The condition

$$\begin{pmatrix} j_{s\infty} \\ j_{t\infty} \end{pmatrix} = \begin{pmatrix} -v_{t\infty} \\ v_{s\infty} \end{pmatrix} \cos\alpha \quad \text{at a perfectly conducting wall} \tag{2.28}$$

follows from equations (2.23), (2.26). These two boundary conditions on an inviscid core flow have not been stated before: they are vital to an understanding of three-dimensional MHD flows. The condition (2.27) may be viewed as a generalization to three dimensions of Shercliff's (1956) result for curved walls of ducts of constant cross-section. He showed how the Hartmann boundary layer provides a relation between the current entering or leaving it and the velocity outside. It is also a corollary to Stewartson's (1960) relation between the vorticity and current content of two-dimensional layers, in describing how the current and vortex lines enter or leave a developing layer. The new idea is that this controls the flow outside. (Equation (2.27) plays the same role as that governing flow into an Ekman layer in rotating fluids.)

3. On the general solution for flow between two non-conducting surfaces

We now use our core-flow solution and boundary-layer matching conditions to examine the flow between two non-conducting surfaces

$$y = F_u(x,z) \quad \text{and} \quad y = F_l(x,z),$$

for example, flow in a duct or flow over a body placed in a duct.

It follows from the boundary condition on $v_{n\infty}$, (2.24), that the zero-order core-flow solution must satisfy the condition

$$v_{n\infty} = 0. \tag{3.1}$$

Using our solution (2.18) we conclude that on $y = F$, where $F = F_u, F_l$,

$$-F\nabla^2 h - g + F_x\left[F\frac{\partial \Psi}{\partial z} - \frac{\partial h}{\partial x} - \frac{\partial \psi}{\partial z}\right] + F_z\left[-F\frac{\partial \Psi}{\partial x} - \frac{\partial h}{\partial z} + \frac{\partial \psi}{\partial x}\right] = 0. \tag{3.2}$$

To calculate the zero-order current density we note that as $M \to \infty$, (2.27) becomes

$$j_{n\infty} = 0, \tag{3.3}$$

whence

$$\Psi + F_x\frac{\partial h}{\partial z} - F_z\frac{\partial h}{\partial x} = 0, \tag{3.4}$$

when $F = F_u, F_l$. Subtracting the two equations in (3.4) we obtain, in general,

$$h = H(D), \quad D(x,z) = F_u - F_l, \tag{3.5}$$

where H is an arbitrary function, and then

$$\Psi = H'(D)\frac{\partial(F_u, F_l)}{\partial(x,z)}. \tag{3.6}$$

Once H is known, (3.2) gives equations for ψ and g, in general.

Two interesting special cases illustrate the general solution (3.5), (3.6). If the duct is symmetric, $F_l = -F_u$, or one wall is perpendicular to $\mathbf{B}_0$, $F_x = F_z = 0$, then

$$\Psi = 0, \quad \text{i.e. } j_y = \partial u/\partial y = \partial w/\partial y = 0. \tag{3.7}$$

It can happen that at some section of such a duct the flow is completely current-free, so that $H \equiv 0$ in the region filled by lines of constant D passing through the section. (This is the case for a constant-area duct, where the current density is of order M^{-1}.) Then the solution of equations (3.2) is

$$g = 0, \quad \psi = S(D). \tag{3.8}$$

Only $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$ are non-zero; and the flow follows lines of constant separation D in the (x,z)-plane.

It would seem that the calculation of the velocity distribution in such a duct (taking into account the Hartmann layers) is a trivial matter if H can be proved zero. However, an examination of the case of a symmetric duct changing from constant to variable area (however gradually) shows that the problem is surprisingly difficult. Singularities appear in the solution (3.8), signalling a complicated type of layer involving both inertial and viscous terms.

We shall return to this question in a later paper: here we restrict our attention to the flow past bodies in parallel-sided ducts.

4. Flow past a thick body in a parallel-sided duct

Consider the flow past a symmetric body, with surfaces $y = \pm f(x,z)$, placed in a parallel-sided duct with non-conducting walls at $y = \pm 1$.† The term thick, as is shown in §6, implies that the y-dimension of the body is $O(1)$ and that the x- and z-dimensions are small compared to N.

The core region is divided into two by the cylinder circumscribing the body with generators parallel to $\mathbf{B}_0$ (see figure 1). First, we treat the interior C_i and the exterior C_e; then the interface layer S between them.

4.1. *Core region* C_e

Consider the general solution (2.17)–(2.19). Unlike the general case discussed in §3, the two conditions (3.4) are not fully effective but lead to the single conclusion

$$\Psi = 0, \quad \text{i.e. } j_y = \partial u/\partial y = \partial w/\partial y = 0. \tag{4.1}$$

† The analysis is easily extended to non-symmetric bodies and to the effects of non-conducting boundaries at $z = \pm b(x)$.

Likewise, the conditions (3.2) are not equations for ψ and g but give

$$g = \frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial z^2} = 0, \quad \text{i.e.} \quad v = 0. \tag{4.2}$$

h itself may be calculated if sufficient information is given about it at large distances and on S. But ψ is left undetermined: the zero-order core equations do not determine the flow uniquely.† Indeed, we shall find that the Hartmann layers control the core flow through the condition (2.27).

As a consequence we have to consider $\mathbf{v}$ and $\mathbf{j}$ as being the leading terms, $\mathbf{v}_0$ and $\mathbf{j}_0$, in an asymptotic expansion, which we tentatively write

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{v}_0+(N^{-1}\mathbf{v}_{1N}+N^{-2}\mathbf{v}_{2N}+\ldots+N^{-r}\mathbf{v}_{rN})+M^{-1}\mathbf{v}_{1M}+\ldots,\\ \mathbf{j} &= \mathbf{j}_0+(N^{-1}\mathbf{j}_{1N}+N^{-2}\mathbf{j}_{2N}+\ldots+N^{-r}\mathbf{j}_{rN})+M^{-1}\mathbf{j}_{1M}+\ldots,\\ p/N &= p_0+(N^{-1}p_{1N}+N^{-2}p_{2N}+\ldots+N^{-r}p_{rN})+M^{-1}p_{1M}+\ldots. \end{aligned}\right\} \tag{4.3}$$

Note that $\mathbf{v}_0 = (u_0(x,z),0,w_0(x,z))$, $\mathbf{j}_0 = (j_{x0}(x,z),0,j_{z0}(x,z))$ and $p_0 = p_0(x,z)$. The series expansion in terms of N^{-1} appears to be the only one which can be constructed in the core where the viscous terms are ignored, as was previously noted by Hunt & Leibovich (1967). The reason we insert the term $M^{-1}\mathbf{v}_{1M}$ is because the boundary conditions are functions of M. Clearly if the series were to be expanded further, we would have to include mixed terms, such as $M^{-1}N^{-r}v_{1MrN}$, and higher-order terms in M^{-1}. When $N^{-r} > M^{-1}$ $(r \geqslant 1)$ each term in the expansion (4.3) is larger than the one to its right, but when $M^{-1} > N^{-1}$ the expansion should be rewritten

$$\mathbf{v} = \mathbf{v}_0+M^{-1}\mathbf{v}_{1M}+N^{-1}\mathbf{v}_{1N}+M^{-1}N^{-1}v_{1M1N}+\ldots, \quad \text{etc.}$$

(Note that, since we assume $R \gg 1$, $M^{-2} \ll N^{-1}$.)

If we neglect terms $O(M^{-2})$ in these expansions then, since the ratio of the viscous terms, $R^{-1}\nabla^2\mathbf{v}$, to the zero-order electromagnetic term $N(\mathbf{j}\times\hat{\mathbf{y}})$ in equation (2.6) is $O(M^{-2})$, we can ignore the former. Thus, the governing equations for the core flow become

$$(\mathbf{v}.\nabla)\mathbf{v} = -\nabla p+N(\mathbf{j}\times\hat{\mathbf{y}}), \tag{4.4}$$

along with (2.7), (2.10), (2.11).

The boundary conditions for the terms we are most interested in, namely those not containing mixed coefficients in N and M, are, in our boundary-layer notation,

$$v_{n_{rN}} = 0, \quad v_{n_{1M}} = \cos\alpha\left[\frac{\partial(v_{s\infty})_0}{\partial s}+\frac{\partial(v_{t\infty})_0}{\partial t}\right], \tag{4.5}$$

and

$$j_{n_{rN}} = 0, \quad j_{n_{1M}} = \frac{\partial(v_{s\infty})_0}{\partial t}-\frac{\partial(v_{t\infty})_0}{\partial s}, \tag{4.6}$$

where $r \geqslant 0$.

† A similar difficulty arises in Ludford's (1961) paper. There it was settled by assuming a charge-relaxation time large compared to transit time of a fluid particle passing the body, whence $\nabla.\mathbf{E} = 0$; which though mathematically acceptable is physically unrealistic. On the other hand, Ludford & Singh (1963) established a unique flow by considering its development in time.

If we now take the first-order term in the expansion (4.3), the curl of (4.4) shows that $\mathbf{j}_{1N}$ satisfies

$$0 = \frac{\partial j_{x_{1N}}}{\partial y}, \quad \mathbf{v}_0.\boldsymbol{\nabla}\left(\frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}\right) = \frac{\partial j_{y_{1N}}}{\partial y}, \quad 0 = \frac{\partial j_{y_{1N}}}{\partial z}. \tag{4.7}$$

Then applying the boundary condition (4.6) to $j_{y_{1N}}$ (which is seen to be linear in y) we find that

$$j_{y_{1N}} = 0; \tag{4.8}$$

hence

$$(\mathbf{v}_0.\boldsymbol{\nabla})\left(\frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}\right) = 0. \tag{4.9}$$

This is the equation of plane rotational flow, and one might conclude that $\mathbf{v}_0$ is determined once the upstream vorticity is specified. (No further restrictions arise from higher-order terms.)

The inference that upstream vorticity can be prescribed is wrong, however. If M^{-1}-terms are considered, we find†

$$\partial \mathbf{j}_{1M}/\partial y = 0, \tag{4.10}$$

so that the boundary condition (4.6), applied at $y = \pm 1$, gives

$$j_{y_{1M}} = \frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x} = 0. \tag{4.11}$$

For flow between parallel walls the Hartmann layers will not accept vorticity; and thus they control the core flow.

Once the upstream conditions are given, in conformity with (4.11), and appropriate matching conditions are provided on S, the flow everywhere in C_e is determined by solving

$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial z^2} = 0, \tag{4.12}$$

where

$$u_0 = -\partial \psi'/\partial z \quad \text{and} \quad w_0 = \partial \psi'/\partial x \tag{4.13}$$

(ψ' is ψ plus the harmonic conjugate of h).

4.2. *Core region C_i*

Here the flow passes between boundaries whose separation varies, namely the walls at $y = \pm 1$ and the body at $y = \pm f(x, z)$. Despite this difference between C_e and C_i, we still need to consider higher-order terms.

Consider first a *non-conducting* body and take $y > 0$. The boundary conditions (3.4) do not degenerate, as they did in C_e, and the general results (3.5), (3.6) apply. With $F_u \equiv 1$ these read

$$h = H(1-f) = \bar{H}(f) \quad \text{and} \quad \Psi = 0, \tag{4.14}$$

so that

$$u_0 = -\bar{H}' f_x - \partial \psi/\partial z, \quad w_0 = -\bar{H}' f_z + \partial \psi/\partial x. \tag{4.15}$$

Before g and ψ can be found from the conditions (3.2) we must determine $\bar{H}$.

It will now be shown that only $\bar{H} \equiv$ const. is acceptable: any other choice gives circulating currents in C_i accompanied by non-zero total mass flux across them.

† As in equation (2.14), since viscous and inertial terms still do not intrude.

(If the body were symmetric about $z = 0$ these would be ruled out immediately.) When $\bar{H}$ varies with f the current lines $f =$ const. are closed contour lines C on the body;† and

$$Q = (1-f)\oint_C (u_0 dz - w_0 dx)$$

is the total flow out of the right cylinder with C as base. When the formulas (4.15) are inserted, Ψ integrates out and we are left with

$$Q = (1-f)\bar{H}'\oint_C (f_z dx - f_x dz).$$

Since C is a line of constant f, this integral is that of $|\mathrm{grad}\, f|$ around C and hence is non-zero. Any variable $\bar{H}$ would therefore give non-zero Q, which is unacceptable. Thus

$$\mathbf{j}_0 = 0. \qquad (4.16)$$

With h constant the boundary conditions (3.2) give

$$g = 0, \quad \psi = \bar{S}(f) \quad (\bar{S} \text{ arbitrary}), \qquad (4.17)$$

as in (3.8); so that

$$v_0 = 0 \qquad (4.18)$$

and the streamlines are parallel to the contours C.

We are left with an undetermined circulatory flow; which makes us expect

$$u_0 = w_0 = 0. \qquad (4.19)$$

Certainly this must be the case if the body is symmetric about $z = 0$. Proof for a general body depends on the controlling action of the Hartmann layers. (Once again the N^{-r}-terms fail us. Since j_{x_1N}, j_{z_1N} can be related to inertial terms in u_0, w_0 it might be thought that the condition of zero current flux through the cylinder on C would lead to $\bar{S} =$ const. But, surprisingly enough, the condition is satisfied for all $\bar{S}$.)

According to equation (4.10) $\mathbf{j}_{1M}$ is a function of x, z only. The boundary condition (4.6), applied on $y = 1$ and $y = f$, shows that it satisfies

$$\left.\begin{aligned} -j_{y_1M} &= \frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}, \\ j_{y_1M} - f_x j_{x_1M} - f_z j_{z_1M} &= \frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}; \end{aligned}\right\} \qquad (4.20)$$

hence, by addition,

$$-f_x j_{x_1M} - f_z j_{z_1M} = 2\left(\frac{\partial u_0}{\partial z} - \frac{\partial w_0}{\partial x}\right).$$

From this we may prove $\bar{S} =$ const. as follows.

The left-hand side is $-\partial(f j_{x_1M})/\partial x - \partial(f j_{z_1M})/\partial z$ since $\partial j_{x_1M}/\partial x + \partial j_{z_1M}/\partial z = 0$. Its integral over the interior of C is therefore zero if there is to be no total current flux through the cylinder on C. The right-hand side then gives

$$\oint_C (u_0 dx + w_0 dz) = -\bar{S}'\oint_C (f_z dx - f_x dz) = 0,$$

i.e. $\bar{S} =$ const.

† For a non-symmetric body the lines of constant f on its top side may not be closed. A more involved argument is then required, which, for simplicity, we omit here.

Thus equations (4.16)–(4.19) hold in C_i. In fact, $\mathbf{j}_{1M} = \mathbf{v}_{1M} = 0$ also.

When the body is *highly conducting*, but the walls remain non-conducting, the condition (3.3) at $y = f$ must be replaced by (2.28) there. We find

$$\Psi = 0, \quad \psi = \text{const.} \tag{4.21}$$

(The latter is most easily obtained by remembering that, when Ψ is zero, ψ is the electric potential.) Both conditions (3.2) apply, and yield

$$(f-1)\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial z^2}\right)+f_x\frac{\partial h}{\partial x}+f_z\frac{\partial h}{\partial z} = 0, \quad g = -\left(\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial z^2}\right). \tag{4.22}$$

Note that there is, in general, flow over such a body, given by

$$u_0 = j_{z0} = -\partial h/\partial x, \quad w_0 = -j_{x0} = -\partial h/\partial z. \tag{4.23}$$

The vortex lines are horizontal: $\partial u_0/\partial z - \partial w_0/\partial x = 0$. Although the analysis to be presented next is adequate in principle to deal with the shear layer surrounding a perfectly conducting body for arbitrary current density in C_e (unlike the non-conducting body as it will turn out), attention will be focused on the simplest case in which the zero-order current density in C_e is zero, so that $h = 0$. C_i is then stagnant and current free.

4.3. *Shear layer S (non-conducting body)*

Since there is no flow within C_i and the velocity distribution in C_e is that of a two-dimensional potential flow, it is not possible for the velocity in C_e to match that in C_i without singular velocity gradients occurring. Therefore, either viscous or inertial terms or both must become as large as the terms retained in the momentum equations, with the consequence that the core flow approximations become invalid. In this section we postulate the existence of a shear layer S lying between C_e and C_i, such that the flow is rendered continuous between these two regions; and by making certain assumptions we find that such a shear layer may be constructed. For the analysis of the layer we take a set of co-ordinate axes (ξ, η, ζ) such that the η-axis is parallel to the y-axis, the ξ-axis is normal to the circumscribing cylinder, and $\xi = 0$, $\eta = 0$ where the body touches this cylinder. The velocities in this set of axes are $(\bar{u}, \bar{v}, \bar{w})$ (see figure 3).

From the solutions in C_e and C_i we know that

$$\bar{v}_e = 0, \quad \mathbf{v}_i = \mathbf{j}_i = 0 \tag{4.24}$$

where the suffices e and i refer to the values of variables on the outer and inner sides of S. We do not know the values of $\bar{u}_e$ and $\bar{w}_e$ from the solution in C_e, but we can state, by hypothesis,

$$\bar{u}_e = O(1), \quad \bar{w}_e = O(1).$$

If we consider the thickness of S to be $\delta\,(\ll 1)$ and write $\xi = X\delta$, then it follows from continuity that

$$\bar{u} = O(1), \quad \bar{w} = O(\delta^{-1}) \quad \text{in } S. \tag{4.25}$$

To calculate $\bar{w}$ take the curl of (2.6). By ignoring higher-order terms in δ we have

$$\frac{1}{\delta}\frac{\partial}{\partial X}\left(\frac{\bar{u}}{\delta}\frac{\partial \bar{w}}{\partial X}+\bar{v}\frac{\partial \bar{w}}{\partial \eta}+\bar{w}\frac{\partial \bar{w}}{\partial \zeta}\right) = N\frac{\partial j_\eta}{\partial \eta}+\frac{1}{R\delta^3}\frac{\partial^3 \bar{w}}{\partial X^3}+O\left(\frac{1}{R\delta^2}\right). \tag{4.26}$$

Differentiating with respect to X and using (2.14) we have

$$\frac{1}{\delta}\frac{\partial^2}{\partial X^2}\left(\frac{\bar{u}}{\delta}\frac{\partial \bar{w}}{\partial X}+\bar{v}\frac{\partial \bar{w}}{\partial \eta}+\bar{w}\frac{\partial \bar{w}}{\partial \zeta}\right) = -\frac{M^2\delta}{R}\frac{\partial^2 \bar{w}}{\partial \eta^2}+\frac{1}{R\delta^3}\frac{\partial^4 \bar{w}}{\partial X^4}+O\left(\frac{1}{R\delta^3}\right). \tag{4.27}$$

There are two possibilities. Balancing the inertial and electromagnetic terms, as was done by Hunt & Leibovich (1967), requires $N\delta^3 = 1$ and $R\delta \gg 1$, or $M^{\frac{1}{2}} \ll R$, but leads to the non-linear equation

$$\frac{\partial^2}{\partial X^2}\left(\bar{u}\frac{\partial \bar{w}}{\partial X}+\delta\bar{v}\frac{\partial \bar{w}}{\partial \eta}+\delta\bar{w}\frac{\partial \bar{w}}{\partial \zeta}\right) = -\frac{\partial^2 \bar{w}}{\partial \eta^2}, \tag{4.28}$$

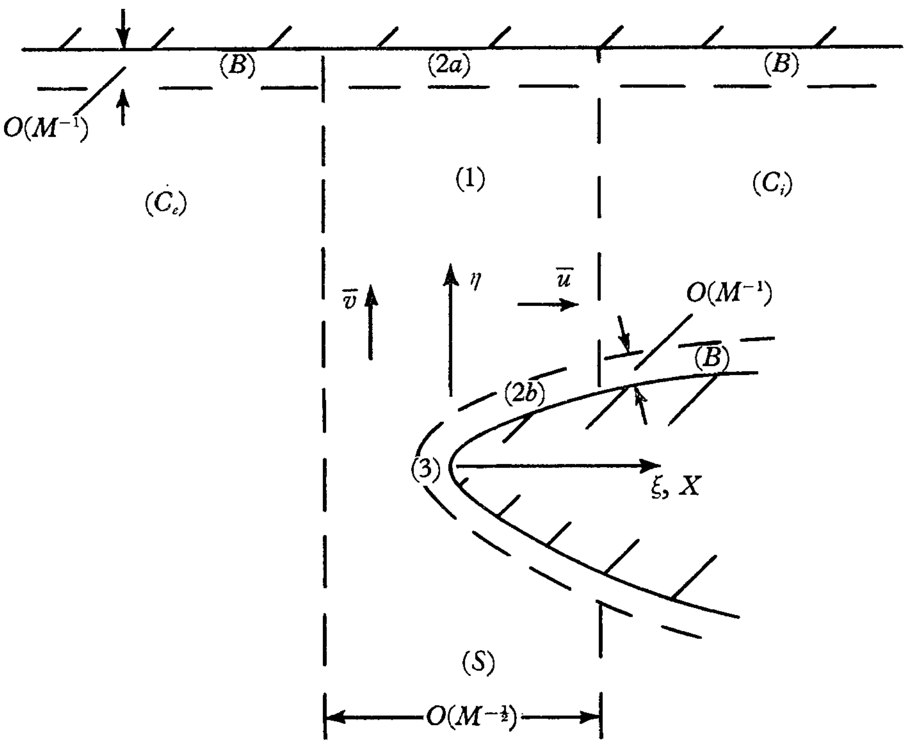

FIGURE 3. Notation for the shear-layer analysis in §4. The body, as assumed in §4, is symmetric.

which we have not been able to solve. On the other hand, a viscous electromagnetic balance, i.e. $M^2\delta^4 = 1$ and $R\delta \ll 1$, leads to the fourth-order linear equation

$$\frac{\partial^4 \bar{w}}{\partial^4 X}-\frac{\partial^2 \bar{w}}{\partial \eta^2} = 0. \tag{4.29}$$

The shear layer in this case has thickness

$$\delta = M^{-\frac{1}{2}} \tag{4.30}$$

in contrast to $N^{-\frac{1}{3}}$ in the other.

There are two disadvantages to confining ourselves to this simpler equation, the first being that it requires the magnetic field to be sufficiently large for

$M^{\frac{1}{2}} \gg R$, a condition difficult to attain experimentally and yet have R sufficiently large for measurements of velocity, etc., to be made. The second and less serious disadvantage is that

$$\mathbf{j}_e = O(M^{-1}), \tag{4.31}$$

as will now be shown.

Suppose $j_{\zeta e}$ is $O(1)$; then, since $j_{\zeta i} = 0$, $\bar{v}$ must be $O(\delta^{-1})$ in the layer according to the induction equation (2.11). If the flow is to follow the body $\bar{u} = O(\delta^{-1})$ also, and then continuity requires $\bar{w} = O(\delta^{-2})$. Thus, whatever the value of M, the inertial terms are large compared to the viscous terms, and we have to treat the full non-linear equation (4.28). $j_{\xi e} = O(1)$ leads to a similar conclusion.

The restriction (4.31) to an inertialess shear layer S is not, however, serious since experiments are usually made with non-conducting side walls, so that far upstream and downstream $j_e = O(M^{-1})$ exactly.

Note that the argument only applies when $\mathbf{j}_i$ is required to be zero. For the highly conducting body and the flat plate considered later the inertialess shear layer is entirely appropriate. Then the conclusion is that $\mathbf{j}$ must be continuous across the layer; this is the condition which cannot be satisfied for the non-conducting body when currents are being driven through C_e.

Now we can formulate our problem precisely. $\mathbf{v}$ and $\mathbf{j}$ are expanded in asymptotic series in $M^{-\frac{1}{2}}$, their forms being justified by consistency.

$$\left.\begin{aligned} \bar{u} &= \bar{u}_0 + M^{-\frac{1}{2}}\bar{u}_1 + \ldots, \\ \bar{v} &= M^{\frac{1}{2}}\bar{v}_0 + \bar{v}_1 + \ldots, \\ \bar{w} &= M^{\frac{1}{2}}\bar{w}_0 + \bar{w}_1 + \ldots; \end{aligned}\right\} \tag{4.32}$$

$$\left.\begin{aligned} j_\xi &= M^{-\frac{1}{2}}j_{\xi_0} + M^{-1}j_{\xi_1} + \ldots, \\ j_\eta &= j_{\eta_0} + M^{-\frac{1}{2}}j_{\eta_1} + \ldots, \\ j_\zeta &= j_{\zeta_0} + M^{-\frac{1}{2}}j_{\zeta_1} + \ldots. \end{aligned}\right\} \tag{4.33}$$

Then equations (2.6), (2.11) give

$$(\partial^4/\partial X^4 - \partial^2/\partial\eta^2)(\bar{w}_0, \bar{w}_1, \bar{v}_0, \bar{v}_1) = 0. \tag{4.34}$$

Having determined $\bar{w}_0, \bar{w}_1, \bar{v}_0, \bar{v}_1$, we may calculate $\bar{u}_0$, $\bar{u}_1$ from (2.7); $j_{\eta_0}, j_{\eta_1}, j_{\zeta_0}, j_{\zeta_1}$ from (2.11); and then j_{ξ_0}, j_{ξ_1} from (2.10).

Now consider the boundary conditions at $\eta = 1$ and $\eta = f(\xi, \zeta)$, where Hartmann-type boundary layers [regions (2*a*) and (2*b*)] occur within the shear layers S. In these layers, since $\partial/\partial n = O(M)$ still dominates $\partial/\partial s$, $\partial/\partial t = O(M^{\frac{1}{2}})$, we can apply (2.27) and find that at

$$\eta = f(\xi, \zeta) = 0 + O(M^{-\frac{1}{2}}), \quad \partial\bar{w}/\partial X = M\delta[-j_\eta + O(M^{-\frac{1}{2}})]$$

for $X > 0$; whence

$$\eta = 0: \left\{\begin{aligned} (\partial^2/\partial X^2 + \partial/\partial\eta)\bar{w}_0 = 0 \quad \text{for} \quad X > 0, \\ \partial\bar{w}_0/\partial\eta = 0 \quad \text{for} \quad X < 0. \end{aligned}\right\} \tag{4.35}$$

The second condition follows by symmetry. At the top wall

$$\partial\bar{w}/\partial X = M\delta[j_\eta + O(M^{-1})];$$

whence $$\eta = 1: (\partial^2/\partial X^2 - \partial/\partial\eta)\bar{w}_0 = 0 \quad \text{for all } X. \tag{4.36}$$

The boundary conditions for $\bar{v}_0$ are, from (2.24),

$$\eta = 0: \begin{cases} \bar{v}_0 = f_\zeta \bar{w}_0 & \text{for} \quad X > 0, \\ \bar{v}_0 = 0 & \text{for} \quad X < 0, \end{cases} \tag{4.37}$$

and

$$\eta = 1: \bar{v}_0 = 0 \quad \text{for all } X. \tag{4.38}$$

Finally, the boundary conditions at $X = \pm\infty$ are

$$X = -\infty: \bar{u}_0 = \bar{u}_e, \quad \bar{v}_0 = \bar{w}_0 = \bar{j}_0 = 0, \tag{4.39a}$$

since otherwise $\bar{w}_e$ would have to be $O(M^{\frac{1}{2}})$ outside the layer, and

$$X = +\infty: \mathbf{v}_0 = \mathbf{j}_0 = 0. \tag{4.39b}$$

The above analysis clearly fails near $X = \eta = 0$ where there are sudden changes in the boundary conditions, but this makes no difference. In general there is a small region of dimensions $O(M^{-1})$ where $\partial/\partial\xi$, $\partial/\partial\eta = O(M)$ and in which the Hartmann-layer conditions are not valid. The governing equations are

$$\left\{\frac{\partial^2}{\partial Y^2} - \left(\frac{\partial^2}{\partial \bar{X}^2} + \frac{\partial^2}{\partial Y^2}\right)^2\right\}(\bar{u}, \bar{w}) = 0 \tag{4.40}$$

when written in terms of the stretched co-ordinates $\bar{X} = M\xi$, $Y = M\eta$.

The general solution of (4.34) for $\bar{w}_0$ consists of two parts: a diffusion in the positive η-direction, which satisfies the boundary condition (4.36) automatically, and a diffusion in the negative η-direction, which this condition shows to be zero in our case. Taking into account the remaining boundary conditions we therefore find

$$\bar{w}_0 = [K(\zeta)/2(\pi\eta)^{\frac{1}{2}}] \exp(-X^2/4\eta). \tag{4.41}$$

[There are many solutions which satisfy the boundary conditions on $\bar{w}_0$ in (4.39), e.g. the X-derivatives of the function (4.41); but only the latter gives a change in $\bar{u}_0$ through the layer. All such additional functions may be excluded for being too singular at $X = \eta = 0$.] The solution for $\bar{v}_0$ is

$$\bar{v}_0 = 0, \tag{4.42}$$

since the $\bar{w}_0$ in (4.37) is now seen to be zero. From these follow in succession

$$\left.\begin{aligned} \bar{u}_0 &= \tfrac{1}{2}K'(\zeta)\operatorname{erfc}(X/2\sqrt{\eta}), \quad \text{so that } K' = \bar{u}_e, \\ \bar{j}_{\eta_0} &= -\frac{K(\zeta)}{4\pi^{\frac{1}{2}}\eta^{\frac{3}{2}}} X \exp(-X^2/4\eta), \quad \bar{j}_{\zeta_0} = 0, \\ \bar{j}_{\xi_0} &= -\frac{K(\zeta)}{4\pi^{\frac{1}{2}}\eta^{\frac{3}{2}}}[X^2/2\eta - 1]\exp(-X^2/4\eta), \end{aligned}\right\} \tag{4.43}$$

when the boundary conditions (4.39) are applied.

The function $K(\zeta)$ is only determined to within an additive constant, since it is $\bar{u}_e = K'$ which is supposed given (by the solution of the potential problem in the next section). With an arbitrary origin for ζ we may write

$$K(\zeta) = c + k(\zeta), \quad k(\zeta) = \int_0^\zeta \bar{u}_e(\zeta')\,d\zeta',$$

where ζ ranges from 0 to l around the circumscribing cylinder of the body. Since the shear layer is not a source or sink of fluid,

$$k(l) = \int_0^l \bar{u}_e(\zeta')\,d\zeta' = 0$$

and K is single valued. To fix c we appeal to the minimum dissipation theorem to be discussed in §5. The dissipation of energy (predominantly in region 3) is proportional to

$$\int_0^l K^2(\zeta')\,d\zeta',$$

which is a minimum for

$$c = -\frac{1}{l}\int_0^l k(\zeta')\,d\zeta'.$$

The corresponding K is independent of the origin of ζ and vanishes at least twice. For a body symmetric about $z = 0$, these zeros give the front and rear lines $z = 0$ on the circumscribing cylinder.

$\bar{w}_1$ and $\bar{v}_1$ satisfy the same boundary conditions (4.35)–(4.38) as $\bar{w}_0$ and $\bar{v}_0$. To be sure, extra terms in $\bar{u}_0$, $\bar{v}_0$, $\bar{w}_0$ and their derivatives do appear in equations (4.35), (4.37) for $X > 0$ (though not (4.36,) (4.38)), but we have just shown these to be zero. Conditions (4.39) also hold for the 1-solution except that

$$\bar{u}_1 = 0 \quad \text{and} \quad \bar{w}_1 = \bar{w}_e \quad \text{at} \quad X = -\infty \tag{4.44}$$

now. The solution will not be written here, since it must be excluded for the reason given next. It is appropriate for a perfectly conducting body, so the formulas will be given there.

The total current in a Hartmann layer is M^{-1} times the velocity just outside, and is directed at right angles to the flow. Hence there is a current $M^{-1}\bar{w}_e$ flowing into the top outside of the shear layer S with nowhere to go but vertically down the layer,† since $\bar{w}_1 = 0$. But having reached the bottom there is still nowhere to go, if we rule out the possibility of an intense equatorial current $O(M\bar{w}_e)$ around the body. Such a current would induce a comparable velocity $\bar{u}$, which is unacceptable to the governing equation (4.40) in region (3). We conclude that

$$\bar{w}_e = 0. \tag{4.45}$$

4.4. *Completion of solution in C_e*

The condition (4.45) reads

$$\partial\psi'/\partial n = 0 \quad \text{on} \quad f(x,z) = 0 \tag{4.46}$$

in terms of the stream function $\psi'(x,z)$ of the flow in C_e (see end of §4.1). The remaining condition is

$$\psi' = u_\infty z \quad \text{at large distances.} \tag{4.47}$$

The streamlines are the force lines for a conducting cylinder $f(x,z) = 0$ placed in a uniform electric field u_∞ in the x-direction.

There is no zero-order current in C_e by hypothesis ($h = 0$). In the Hartmann layers the current flows at right angles to the streamlines, in the negative z-direction at both walls.

† This result may be checked by integrating the j_{η_1} of (4.51) across the layer.

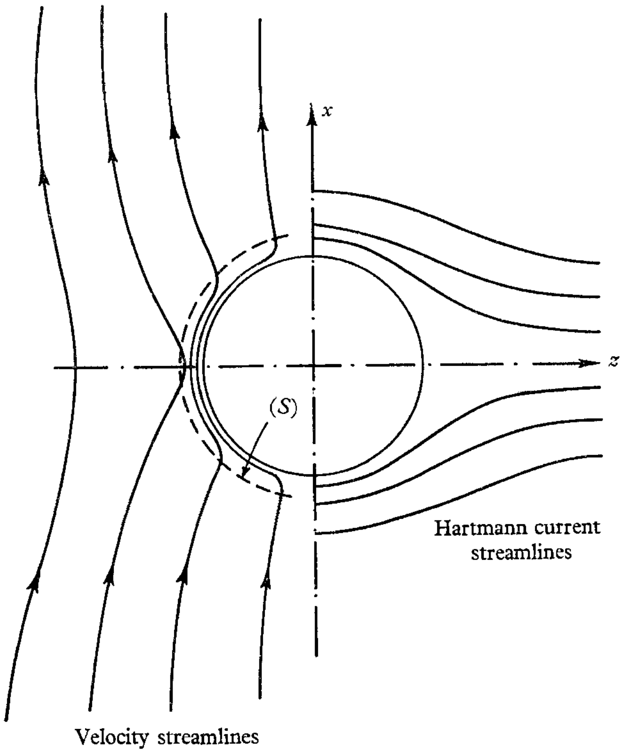

FIGURE 4. Streamlines and current lines for flow over a non-conducting sphere when $\mathbf{j}_e = O(M^{-1})$.

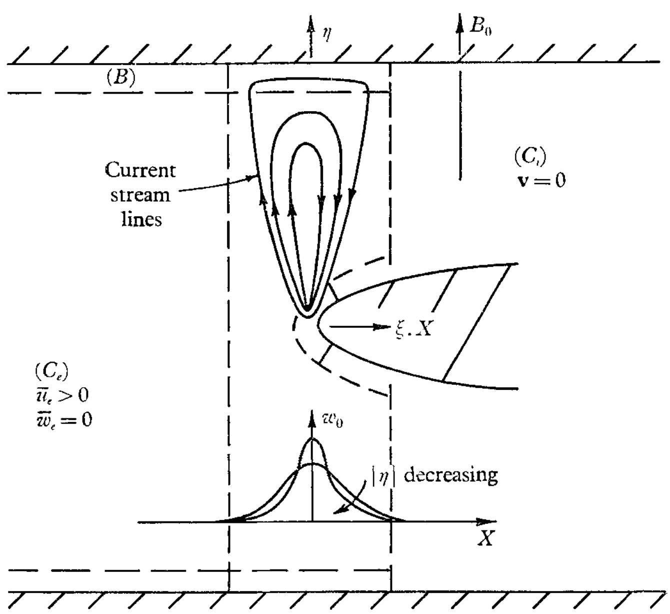

FIGURE 5. Flow in the shear layer for a thick non-conducting body: current stream lines and graph of $\bar{w}_0$ at constant values of η.

Figure 4 shows the streamlines and $O(M^{-1})$ current lines for a non-conducting sphere. In figure 5 we sketch the current lines of $\mathbf{j}_0$ and a graph of w_0 in the shear layer S.

4.5. *Shear layer for a highly conducting body*

As was pointed out in §4.3, our shear-layer analysis applies here even when there are $O(1)$ currents in C_e. An $O(1)$ term must be added to j_ξ in the expansions (4.33). Not surprisingly it turns out to be independent of ξ. Similarly, terms independent of ξ may be added to j_{η_0} and j_{ζ_0}. All this means is that any $O(1)$ core current must be continuous across the layer, and does not affect its dynamics. However, (4.22) is so difficult to solve that we restrict attention to

$$g = h = 0, \quad \text{i.e. } \mathbf{v} = \mathbf{j} = 0 \quad \text{in} \quad C_i, \tag{4.48}$$

which implies zero $O(1)$ current in C_e, the case of greatest practical interest.

In analysing the layer S the only change is the boundary condition on the body, where (2.28) must be used in place of (2.27); but the same solutions $\overline{w}_0$ and $\overline{w}_1$ result. However, this time the currents for $\overline{w}_1$ can flow along the surface of the body when they reach the bottom of S; and an examination of region (3) shows that we must now take

$$\overline{u}_e = 0 \tag{4.49}$$

so that $\overline{w}_0 = 0$.

The solution in the shear layer S is therefore

$$\overline{w}_1 = \tfrac{1}{2}\overline{w}_e \operatorname{erfc}(X/2\sqrt{\eta}), \quad \overline{v}_1 = 0; \tag{4.50}$$

from which follow

$$\left.\begin{aligned} \overline{u}_1 &= \frac{d\overline{w}_e}{d\zeta}[\sqrt{(\eta/\pi)}\exp(-X^2/4\eta) - (X/2)\operatorname{erfc}(X/2\sqrt{\eta})], \\ j_{\eta_1} &= -\frac{\overline{w}_e}{2\sqrt{(\pi\eta)}}\exp(-X^2/4\eta), \quad j_{\zeta_1} = 0, \\ j_{\xi_1} &= -\frac{\overline{w}_e}{4\sqrt{\pi}\,\eta^{\frac{3}{2}}}X\exp(-X^2/4\eta). \end{aligned}\right\} \tag{4.51}$$

5. Flow over a flat plate

Consider a non-conducting flat plate of arbitrary shape $z = C(x)$ located in the centre plane of the parallel-sided channel (figure 6). C_i is now similar to C_e; in particular, there will be flow there also. The link between the two flows is provided by the shear layer S, which we treat as before.

If we consider the first-order, 1-solution we find that $\overline{w}_1 \neq 0$ since no singularity need appear in region (3). The explanation is that part of the current $\overline{w}_e/M$ goes through the top of the shear layer into the Hartmann layer between C_i and the top wall; while the rest flows down the shear layer and out of the bottom into the Hartmann layer on top of the plate, as shown in figure 6. Since each of these Hartmann layers has a current $\overline{w}_i/M$ flowing into it, we must have

$$\overline{w}_e = 2\overline{w}_i. \tag{5.1}$$

To the 1-solution satisfying this condition (in place of $\overline{w}_i = 0$) we may add a zero order, 0-solution effecting the jump from arbitrary $\overline{u}_e$ to arbitrary $\overline{u}_i$. [In equation (4.43) K' is changed to $\overline{u}_e - \overline{u}_i$ and $\overline{u}_i$ is added.] All such solutions are acceptable to the region (3), so that we have a basic indeterminacy.

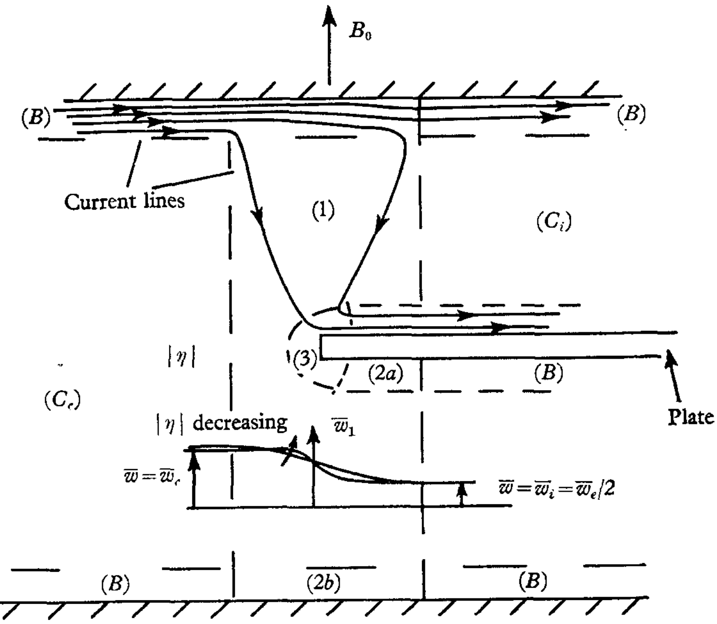

FIGURE 6. Flow in the shear layer for a flat plate: current stream lines, showing how current enters and leaves the boundary layers (*B*) via the shear layers (*S*), and graph of $\overline{w}_1$ at constant values of η.

A choice can be made on the basis of energy dissipation. If the 0-solution is not

$$\overline{u}_0 = \overline{u}_e = \overline{u}_i, \tag{5.2}$$

circulating currents (figure 5) act in the shear layer to increase the dissipation; i.e. (5.2) provides minimum energy loss and hence may be expected on physical grounds. Support for this choice is provided by a general result on inertialess MHD flows due to Moffatt (but unpublished): amongst all flow and current fields satisfying boundary conditions and continuity, the solution is the one with least dissipation. (The corresponding hydrodynamic result, due to Helmholtz and Korteweg, appears in Lamb (1932).) If one has confidence in asymptotic methods, it must be that all choices except (5.2) lead to contradictions (other than in boundary conditions and continuity) at a later stage in the approximation.

The 1-solution (4.50, 4.51) must be modified to accommodate the different condition (5.1). Thus

$$\left.\begin{aligned} \overline{w}_1 &= \frac{\overline{w}_e}{4}[2+\operatorname{erfc}(X/2\sqrt{\eta})], \\ \overline{u}_1 &= \frac{1}{4}\frac{d\overline{w}_e}{d\zeta}[2(\eta/\pi)^{\frac{1}{2}}\exp(-X^2/4\eta) - X\{2+\operatorname{erfc}(X/2\sqrt{\eta})\}], \end{aligned}\right\} \tag{5.3}$$

while $\overline{w}_e$ is replaced by $\frac{1}{2}\overline{w}_e$ in j_{η_1} and j_{ξ_1}. Apart from u_0 only

$$j_{\zeta_0} = j_{\zeta_i}, \quad j_{\xi_0} = -\frac{dj_{\zeta_e}}{d\zeta} X$$

are non-zero for the 0-solution. This is in accordance with the remarks at the beginning of §4.5, which also allow an $O(1)$ term to be added in j_ξ provided it is constant across the layer. In short,

$$\mathbf{j}_e = \mathbf{j}_i. \tag{5.4}$$

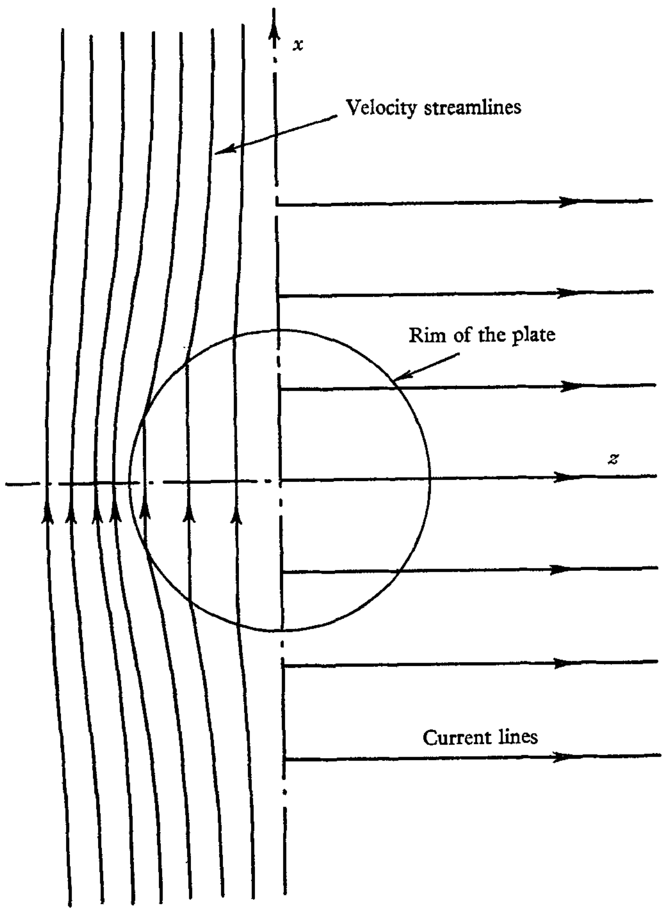

FIGURE 7. Streamlines and current lines for flow over a non-conducting circular plate.

The solution in C_e and C_i is now easily completed. Conditions (5.1), (5.2) read

$$(\partial\psi'/\partial n)_e = 2(\partial\psi'/\partial n)_i, \quad \psi'_e = \psi'_i \quad \text{at} \quad z = C(x) \tag{5.5}$$

in terms of the stream function $\psi'(x, z)$. (See end of §4.1 which now applies to C_i also.) The remaining condition is

$$\psi' = u_\infty z \quad \text{at large distances.} \tag{5.6}$$

The streamlines are the induction lines for a cylinder $z = C(x)$, of permeability $\frac{1}{2}$ that of its surroundings, placed in a uniform magnetic field u_∞ in the x-direction.

Since the current is continuous across $z = C(x)$, whatever is applied uniformly at infinity will pass through undisturbed.

5.1. *Example: elliptical plate*

Consider an ellipitically shaped plate with semi-axes m, n parallel and perpendicular to the flow. The solution is

$$\left.\begin{aligned} u &= u_\infty(1+\phi_x), \quad w = u_\infty \phi_z \quad \text{in} \quad C_e, \\ u &= \alpha u_\infty, \quad w = 0 \quad \text{in} \quad C_i, \end{aligned}\right\} \tag{5.7}$$

where
$$\phi = \frac{mn}{2m+n}\left(\frac{m+n}{|m-n|}\right)^{\frac{1}{2}} e^{-\lambda}\cos\mu, \quad \alpha = \frac{m+n}{2m+n} \tag{5.7a}$$

and
$$x = |m^2-n^2|^{\frac{1}{2}} \begin{matrix}\cosh\\ \sinh\end{matrix}\, \lambda\cos\mu, \quad y = |m^2-n^2|^{\frac{1}{2}} \begin{matrix}\sinh\\ \cosh\end{matrix}\, \lambda \sin\mu \tag{5.7b}$$

according as $m \gtrless n$.

For this particular shape the flow in C_i, i.e. above and below the plate, is uniform and parallel to the incident stream, but is reduced to a fraction α of that stream. As m/n increases from 0 through 1 (circle illustrated in figure 7) to ∞, α decreases from 1 through $\frac{2}{3}$ to $\frac{1}{2}$.

These results are of great significance. When the plate is sufficiently short in the flow direction (m fixed, $n \to \infty$) a *two-dimensional* description of the flow is accurate and the analysis of Dix (1963) is appropriate. When the plate is sufficiently long in the flow direction ($m \to \infty$, n fixed) the asymptotic state proposed by Hasimoto (1960) develops, with $u = \frac{1}{2}u_\infty$ above and below the plate.

6. Remarks and conclusions

The methods used in §§4, 5, and described in some detail, will solve other three-dimensional flow problems. In a future paper we shall treat various kinds of diverging ducts. However, it is of interest here to make some remarks about flows over other kinds of body and in ducts whose walls are not parallel to each other.

(i) Let an arbitrary, thick, non-conducting body (e.g. an ellipsoid with its axis at an angle to the stream) be placed anywhere in a parallel-sided duct. Then $\mathbf{v} = 0$ in C_i, as before, and the flow in C_e is given by (4.46), (4.47), where $f(x,z) = 0$ is now the circumscribing cylinder of the body.

(ii) If the duct is described by $y = \pm F(x)$ with $F' \neq 0$, and a non-conducting body is placed in it, then $\mathbf{v} = 0$ in C_i and the flow in C_e is determined by S.

(iii) On the other hand, if $y = \pm F(z)$ with $F' \neq 0$, the flow in C_e is determined, and we find $\mathbf{v}$ is not zero in C_i but is determined by S. The complete analysis of this problem is very complicated and we have not attempted it.

(iv) In the completely general case $y = F(x,z)$, where $\partial F/\partial x \neq 0$ and $\partial F/\partial z \neq 0$, we have not even solved the duct flow problem by itself, let alone with a body in it. However, we can state that $\mathbf{v}$ is not necessarily zero in C_i, and that the flows in C_e and C_i depend on each other and S.

One of the interesting results of our three-dimensional analysis is that it demonstrates under what circumstances a two-dimensional analysis is valid. This is connected with making the term 'thick' more precise, which we shall now do.

Consider the flow over a non-conducting body of thickness t with length m in the x-direction and length n in the z-direction, placed in a parallel-sided duct. Then, whatever the ratios m/n, the body will be thick and inhibit flow over itself when

$$\text{both} \quad t/m \quad \text{and} \quad t/n \gg N^{-1}. \tag{6.1}$$

On the other hand, when

$$\text{both} \quad t/m \quad \text{and} \quad t/n \sim N^{-1} \text{ or smaller,} \tag{6.2}$$

the body behaves like a flat plate (§5). To see this, note that the only boundary conditions which were sensitive to the thickness of the body are (3.4): for a thin body (plate) they degenerate into the single condition $\Psi = 0$. When both the representative slopes t/m, t/n on the body are large compared to N^{-1} both F_x- and F_z-terms appear and the thick-body analysis applies. In case (6.2) these terms are relegated to the N^{-1}-conditions and the flat-plate analysis is appropriate.

The flow will become two-dimensional in the sense $\partial/\partial z = 0$ as $m/n \rightarrow 0$ when

$$t/n \sim N^{-1}. \tag{6.3}$$

For then F_z drops out of (3.4) and the solution at each section $z = z_0$ is that of Hunt & Leibovich (1967) for plane flow between walls $y = 1$, $F(x, z_0)$, the function F varying slowly with z_0. However, the Hunt–Leibovich solution must be used with caution. It applies accurately to the more realistic case of a non-conducting cylinder which spans the duct in the z-direction from one conducting side wall to the other. But any bulging of the cylinder in the middle will produce closed contour lines, and the flow will be blocked.

Similarly, the flow becomes two-dimensional in the sense $\partial/\partial x = 0$ as $m/n \rightarrow \infty$ when

$$t/m \sim N^{-1}. \tag{6.4}$$

It follows that Hasimoto's (1960) analysis based on the assumption $\partial/\partial x = 0$, can only describe flows over finite bodies for a particular range of N, namely, $t/m \sim N^{-1} \ll 1$. (A similar remark applies to the Hunt–Leibovich analysis.)

The results of §4 that, if the zero-order current is zero far from the body, there can be no flow over either a highly conducting or non-conducting body, should be compared with those of Ludford & Singh (1963). They examined unconfined flow over a sphere, assuming zero viscosity, $N \gg 1$, and R_m arbitrary. In order to render the steady solution determinate the transient problem was considered. They found the velocity over a non-conducting sphere to be $0{\cdot}38u_\infty$ and over a perfectly conducting sphere to be zero. Different results were to be expected not only because one flow is confined and the other is not, but because the Hartmann layers, which we have shown to control the flow, were neglected by Ludford & Singh.

We mentioned in §2 some of the similarities and differences in the equations for non-conducting flow in a rotating environment and our MHD flows. Later sections show how these mathematical similarities lead to physical similarities. We saw in §3 how, when $\mathbf{j}_0 = 0$, the flow follows lines of constant separation

$D = (F_u - F_l)$. Exactly the same occurs in a rotating fluid because the length of vortex lines has to remain constant. In §4.2 we saw how $\mathbf{v} = 0$ in C_i for a non-conducting body; the flow must follow lines of constant f and viscosity necessitates such a flow being zero. Exactly the same analysis arises in rotating fluids in that, if we can show $\mathbf{j}_0 = 0$, the Taylor–Proudman equation $\partial \mathbf{v}/\partial y = 0$ holds (see (2.11)); and that Hartmann layers play the same role as Ekman layers.

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